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Twistor Diagrams for Spinning Massless Free Fields

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A set of simple identities is utilized to build up new projective twistor diagrams for massless free fields of arbitrary spin in real Minkowski space. It is effectively shown that the inner structure of the configurations which arise out of implementing the relevant techniques has characteristics that are different from those of the conventional diagrams associated with the Kirchoff–D'Adhemar–Penrose integral expressions. A nonhomogeneous version of the configurations is also provided.

1. INTRODUCTION

One of the usual devices for evaluating explicitly massless free fields of arbitrary spin in both flat and curved spacetimes is constituted by the so-called Kirchoff–D'Adhemar–Penrose (KAP) integral expressions (Penrose, 1980; Penrose and Rindler, 1984). Normally, such expressions carry integrands that show up as two-forms set upon spacelike two-spheres arising from intersections between null hypersurfaces of real Minkowski space RM and the past null cones of the points at which the fields are to be actually calculated. Accordingly, the data which generate the fields appear as the result of the action of certain conformally invariant operators on elementary initial data centered at points belonging to the null hypersurfaces.

A remarkable feature of the KAP expressions is the fact that they neatly fit in with the twistor formalism (Cardoso, 1991). In fact, the basic procedures leading to the standard twistorial version of the integrals include taking the future null cone C_O^+ of some origin O of RM as the initial-datum hypersurface for the fields. Thus, the RM integrals can be thought of as being taken over (compact) spaces of two-edge null zigzags that start at O and terminate at fixed points lying in the (convex) interior of the closure of C_O^+. What particu-

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larly arises out of completing the transcription procedures is that the conformal symmetry borne by the underlying Minkowskian structures is reduced to the Poincaré symmetry. The crucial point is that the S^2-spheres turn out to be split up into products of projective S^1-structures when the relevant prescriptions are effectively worked out, whence, in the case of either handedness, the overall contours occurring in the respective twistor integrals bear the topology $S^1 \times S^1$, likewise being contained in products of subsets of the Riemann spheres, which appropriately correspond to the projective lines associated with the starting and ending vertices of the null graphical configurations. It appears that the appropriateness referred to above is essentially related to the handedness-valence hypothesis arising in the framework of twistor theory (Penrose and MacCallum, 1972). In addition, it has been shown that the latter integrals can be reexpressed in terms of homogeneous twistor diagrams by making use of a method which enables one to pass directly from integrals of holomorphic projective one-forms to integrals of holomorphic projective three-forms (Cardoso, 1993). Each of the resulting configurations thus carries one of the universal projective twistor-vertex three-forms along with a pair of appropriate constant twistors.

In the present paper, we utilize a set of simple identities to build up projective two-spotted-vertex twistor diagrams for the KAP integrals. Roughly speaking, the main idea here is to prescribe the functional dependence of the twistor data for the fields in such a way that the coupling of the conventional (holomorphic) twistor-datum one-forms with certain normalized auxiliary integrals automatically yields Poincaré-invariant integrands which do not involve derivatives of the twistor-function kernels explicitly. The auxiliary structures are taken as one-dimensional projective integrals which carry only infinity-twistor pieces involving appropriately the (independent) twistors that enter into the definition of the initial data on C_O^+. Subsequently, we will make use of trivial differential relations to convert the new diagrams into inhomogeneous structures. For the sake of completeness, we will first recall the twistor integrals mentioned before (Section 2). The construction of our configurations is carried out in Section 3. There, it will suffice to carry through only the procedures that give rise to the diagrams for unprimed fields. Also, it will be assumed at the outset that the fields carry positive energy. This assumption will indeed facilitate setting up the key blocks in a natural way. We make some general remarks on the pertinent techniques in Section 4.

One of the motivations lying behind the completion of our procedures rests upon the belief that the corresponding methods would eventually provide fresh insights into the twistor description of massless free fields. Throughout the work, we will employ the natural system of units wherein $c = \hbar = 1$ together with the notational devices provided by Penrose and Rindler (1984, 1986). The indices which conventionally label the coordinates of spacetime points will be suppressed.

2. STANDARD INTEGRALS

The standard twistor KAP integral for a left-handed massless free field $\phi_{AB\ldots C}(x)$ of spin $s < 0$ on RM is written as (Cardoso, 1991)

$$\phi_{AB\ldots C}(\overset{2}{x}) = \frac{(-1)^{-2s+1}}{2\pi i} \oint_{\Gamma_1\times\Gamma_2} \overset{2}{o}_A \overset{2}{o}_B \ldots \overset{2}{o}_C \underset{\sim}{\Phi}(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) \wedge \delta\overset{2}{W} \tag{2.1}$$

where the point $\overset{2}{x}$ is taken to be future-null-separated from some point $\overset{1}{x} \in C_O^+$, the spinor $\overset{2}{o}_A$ being chosen covariantly constant along the (null) geodesic that passes through $\overset{1}{x}$ and $\overset{2}{x}$, and $\delta\overset{2}{W}$ stands for $I^{\mu\nu}\overset{2}{W}_\mu\, d\overset{2}{W}_\nu$. The W-twistors are null and satisfy the (conjugate) incidence conditions at $\overset{k}{x}$, being explicitly defined as $\overset{k}{W}_\alpha = (\overset{k}{o}_A, \overset{k}{W}^{A'})$, with the label k taking either the value 1 or 2. In fact, the twistor $\overset{1}{W}_\alpha$ is associated with the generator γ_1 of C_O^+, which contains $\overset{1}{x}$, its π-part being also covariantly constant along γ_1. The quantity $\underset{\sim}{\Phi}(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha)$ is the twistor-datum one-form for the field under consideration. It actually bears holomorphicity in both variables, and is defined on the product space $\mathfrak{D}_0^* \times \mathfrak{D}_2^*$, which involves two (closed) subsets of the Riemann spheres associated to the dual projective lines corresponding to O and $\overset{2}{x}$. Its defining expression is written out explicitly as

$$\underset{\sim}{\Phi}(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) = -\frac{\partial}{\partial \overset{2}{W}_\mu} \Phi(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha)\, d\overset{1}{W}_\mu \tag{2.2}$$

with the function kernel borne by the right-hand side amounting to the twistor version of the null datum on C_O^+ for the field. It also has the symmetry-homogeneity property

$$\underset{\sim}{\Phi}(a\overset{1}{W}_\alpha, b\overset{1}{W}_\alpha + c\overset{2}{W}_\alpha) = c^{2s-2}\underset{\sim}{\Phi}(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) \tag{2.3}$$

where a and c belong to C $-$ $\{0\}$ and $b \in$ C, with C standing for the set of complex numbers. Whence the integral (2.1) really carries a scaling-invariant (SI) integrand. Each of the contours Γ_k is an S^1-contour subject to the simple prescription

$$\mathfrak{D}_0^* \supset \Gamma_1 \simeq S^1, \qquad \mathfrak{D}_2^* \supset \Gamma_2 \simeq S^1 \tag{2.4}$$

We stress that a typical twistor function which occurs in the universal contour integral for the field (see, for instance, Penrose and Rindler, 1986) turns out to be formally expressed as

$$f_s(\overset{2}{W}_\alpha) = (-1)^{-2s+1} \oint_{\Gamma_1} \underset{\sim}{\Phi}(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) \tag{2.5}$$

The projective one-spotted-vertex diagrammatic representation of the field is readily accomplished by inserting into the integrand of (2.1) the SI conformally invariant two-form (Cardoso, 1993)

$$\underset{\sim}{\Theta}(\overset{2}{W}_\alpha) = \frac{1}{(2\pi i)^2} \frac{d(A^\mu \overset{2}{W}_\mu)}{(A^\mu \overset{2}{W}_\mu)} \wedge \frac{D(B^\nu \overset{2}{W}_\nu)}{(B^\nu \overset{2}{W}_\nu)} \tag{2.6}$$

where A^μ and B^ν are fixed null twistors through an arbitrary point $y \in$ RM. We thus have the SI integral

$$\phi_{AB\ldots C}(\overset{2}{x}) = \frac{1}{(2\pi i)^3} \oint_{\mathfrak{C}_2} \overset{2}{\delta}_A \overset{2}{\delta}_B \ldots \overset{2}{\delta}_C \frac{(I_{\mu\nu} A^\mu B^\nu) f_s(\overset{2}{W}_\alpha) \Delta \overset{2}{W}}{(A^\mu \overset{2}{W}_\mu)(B^\nu \overset{2}{W}_\nu)} \tag{2.7}$$

where

$$\Delta \overset{2}{W} = \frac{1}{3!} \epsilon^{\alpha\beta\gamma\delta} \overset{2}{W}_\alpha \, d\overset{2}{W}_\beta \wedge d\overset{2}{W}_\gamma \wedge d\overset{2}{W}_\delta$$

and $\mathfrak{C}_2$ is a suitable three-real-dimensional contour lying in $\mathcal{A} \times \mathcal{B} \times \mathfrak{D}_0^* \times \mathfrak{D}_2^*$ whose topology is $S^1 \times S^1 \times S^1$, with $\mathcal{A}$ and $\mathcal{B}$ being the "unstarred" projective versions of A^μ and B^ν, respectively. Indeed, each of the sets $\mathcal{A}$ and $\mathcal{B}$ consists of a single point belonging to the projective line associated with y. Consequently, we are led to the diagrammatic structures as displayed in Cardoso (1993).

3. EXPLICIT TWISTOR DIAGRAMS

Let us consider the integral

$$\mathfrak{I}(\overset{2}{W}_\alpha) = \frac{1}{2\pi i} \oint_{\mathscr{C}_1} \frac{(I^{\alpha\beta} J_\alpha Y_\beta)(I^{\rho\sigma} S_\rho R_\sigma)(I^{\mu\nu} \overset{2}{W}_\mu \, d\overset{1}{W}_\nu)}{(I^{\alpha\beta} \overset{1}{W}_\alpha R_\beta)(I^{\rho\sigma} \overset{1}{W}_\rho S_\sigma)} \tag{3.1}$$

where $\mathscr{C}_1$ is a one-real-dimensional contour, and S_ρ, R_σ are arbitrary (constant) null twistors. The pertinent prescription will be made up briefly. Hence, parametrizing $\overset{1}{W}_\alpha = J_\alpha + \zeta Y_\alpha$, with $\zeta \in C - \{0\}$, and invoking the independence between the W-twistors yields

$$\mathfrak{I}(\overset{2}{W}_\alpha) = \frac{(I^{\alpha\beta} J_\alpha Y_\beta)(I^{\rho\sigma} S_\rho R_\sigma)(I^{\mu\nu} \overset{2}{W}_\mu Y_\nu)}{(I^{\alpha\beta} Y_\alpha R_\beta)(I^{\rho\sigma} Y_\rho S_\sigma)} \frac{1}{2\pi i} \oint_{\mathscr{C}_1} \frac{d\zeta}{(\zeta - a)(\zeta - b)} \tag{3.2}$$

where

$$a = -(I^{\alpha\beta} J_\alpha R_\beta)(I^{\lambda\tau} Y_\lambda R_\tau)^{-1}, \qquad b = -(I^{\alpha\beta} J_\alpha S_\beta)(I^{\lambda\tau} Y_\lambda S_\tau)^{-1}$$

Obviously, the nullity of $\overset{1}{W}_\alpha$ implies that J_α and Y_α are also incident null twistors. In order to obtain a useful result, we take $\mathscr{C}_1$ as an oriented circled loop which lies in the ζ-plane and surrounds either a or b, supposing additionally that the flag poles associated with the (null) geodesics corresponding to Y_α, R_β and Y_α, S_β are *not* in each case proportional to one another. Therefore, choosing the contour orientations in an appropriate way and making use of the relation

$$(I^{\alpha\beta}Y_\alpha R_\beta)(I^{\lambda\tau}J_\lambda S_\tau) - (I^{\alpha\beta}Y_\alpha S_\beta)(I^{\lambda\tau}J_\lambda R_\tau) = (I^{\alpha\beta}J_\alpha Y_\beta)(I^{\lambda\tau}S_\lambda R_\tau) \quad (3.3)$$

we obtain the (SI) normalized integral

$$\mathcal{J}(\overset{2}{W}_\alpha) = (I^{\mu\nu}\overset{2}{W}_\mu Y_\nu)^{-1}\mathfrak{I}(\overset{2}{W}_\alpha) = 1 \quad (3.4)$$

At this point, we effectively assume that the strongly required skew symmetry of $\Phi(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha)$ is prescribed as

$$\Phi(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) = \Phi(\overset{1}{W}_\alpha, I^{\mu\nu}\overset{1}{W}_\mu \overset{2}{W}_\nu) \quad (3.5)$$

whence one of the homogeneity properties of the Φ-function (Penrose, 1975) turns out to be "reduced" to the statement

$$\overset{2}{W}_0 \frac{\partial}{\partial \overset{2}{W}_0} \Phi(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) + \overset{2}{W}_1 \frac{\partial}{\partial \overset{2}{W}_1} \Phi(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) = (2s - 1)\Phi(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) \quad (3.6)$$

Some elementary computations thus give rise to the wedge-product relation

$$\underset{\sim}{\Phi}(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) \wedge (I^{\mu\nu}\overset{2}{W}_\mu \, d\overset{1}{W}_\nu) = (\tfrac{1}{2} - s)\Phi(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) I^{\mu\nu} \, d\overset{1}{W}_\mu \wedge d\overset{1}{W}_\nu \quad (3.7)$$

Furthermore, inserting into the integrand of (2.1) the structure (3.1) together with the result of the combination of (3.7) with the trivial identity

$$\frac{1}{2} I^{\mu\nu} \, d\overset{1}{W}_\mu \wedge d\overset{1}{W}_\nu = \frac{d\overset{1}{W}_0}{\overset{1}{W}_0} \wedge \delta\overset{1}{W} \quad (3.8)$$

and performing the $\overset{1}{W}_0$-integral leads to the SI expression

$$\phi_{AB\ldots C}(\overset{2}{x}) = \frac{1}{(2\pi i)^2} \oint_{\gamma_{12}} \overset{2}{\delta}_A \overset{2}{\delta}_B \ldots \overset{2}{\delta}_C F_s(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) \delta\overset{1}{W} \wedge \delta\overset{2}{W} \quad (3.9)$$

where the twistor function $F_s(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha)$ accordingly carries a suitable $2\pi i$ factor and takes up $\Phi(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha)$ along with the product of $(I^{\mu\nu}Y_\mu \overset{2}{W}_\nu)^{-1}$ with the infinity-twistor factors that occur explicitly in the integrand of (3.1), thereby bearing the spin-weight character $\{-2, 2s - 2; 0, 0\}$. The contour γ_{12} appears as the product of two appropriate one-real-dimensional contours having the same topology as before. It should be pointed out that (3.8) still

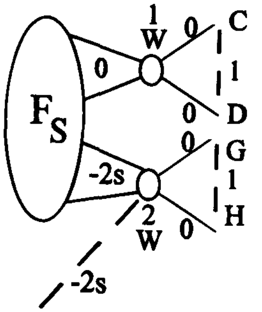

Fig. 1. A projective two-spotted-vertex twistor diagram associated with the KAP integral for a left-handed massless free field of spin $s < 0$. All the lines and numbers bear the usual meaning.

holds when the component $\overset{1}{W}_0$ is replaced by $\overset{1}{W}_1$. Now, using the method leading to the expression (2.7) yields

$$\phi_{AB\ldots C}(\overset{2}{x}) = \frac{1}{(2\pi i)^6} \oint_{\Gamma_{12}} \frac{\overset{2}{\partial}_A \overset{2}{\partial}_B \ldots \overset{2}{\partial}_C (I_{\mu\nu} C^\mu D^\nu)(I_{\beta\gamma} G^\beta H^\gamma)}{(C^\beta \overset{1}{W}_\beta)(D^\gamma \overset{1}{W}_\gamma)(G^\mu \overset{2}{W}_\mu)(H^\nu \overset{2}{W}_\nu)} \times F_s(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha)\, \Delta \overset{1}{W} \wedge \Delta \overset{2}{W} \tag{3.10}$$

with the auxiliary twistors and either of the contour structures involved being specified in a way similar to that given by (2.7). We are thus led to the diagram depicted in Fig. 1.

It is evident that at this stage we are able to write down explicit diagrammatic equalities involving the above structure and the ones for left-handed fields provided in Cardoso (1993). Additionally, we can immediately construct the corresponding diagrams for right-handed fields by taking complex conjugations and changing kernel letters appropriately. We will make a further point concerning such correspondences in Section 4. A relevant equality is exhibited in Fig. 2.

A nonhomogeneous two-spotted-vertex diagram for the field with which we have effectively been dealing can at once be built up from Fig. 1 by employing the differential relation

$$\frac{d\xi_{(k)}}{\xi_{(k)}} \wedge \Delta \overset{k}{W} = d^4 \overset{k}{W} \tag{3.11}$$

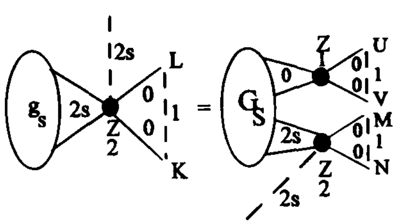

Fig. 2. A projective diagrammatic equality for a right-handed massless free field of spin $s > 0$. The twistor functions and vertices occurring in the structures are all specified in a way similar to that for the left-handed case.

where $\xi_{(k)}$ denotes any (nonvanishing) component of $\overset{k}{W}_\alpha$, and

$$d^4\overset{k}{W} = \frac{1}{4!}\,\epsilon^{\alpha\beta\gamma\delta}\, d\overset{k}{W}_\alpha \wedge d\overset{k}{W}_\beta \wedge d\overset{k}{W}_\gamma \wedge d\overset{k}{W}_\delta \tag{3.12}$$

is the usual conformally invariant twistor-vertex four-form (Penrose, 1975). Thus, the integral (3.10) gets replaced by

$$\phi_{AB\ldots C}(\overset{2}{x}) = \frac{1}{(2\pi i)^8}\oint_{C_{12}} \frac{\overset{2}{\partial}_A\overset{2}{\partial}_B\ldots\overset{2}{\partial}_C(I_{\mu\nu}C^\mu D^\nu)(I_{\beta\gamma}G^\beta H^\gamma)}{(C^\beta\overset{1}{W}_\beta)(D^\gamma\overset{1}{W}_\gamma)(G^\mu\overset{2}{W}_\mu)(H^\nu\overset{2}{W}_\nu)} \times F_s(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha)\, d^4\overset{1}{W} \wedge d^4\overset{2}{W} \tag{3.13}$$

with C_{12} being a (compact) eight-real-dimensional contour given as the product of S^1-structures. We notice that the implementation of this latter prescription does not affect the singularity-line configuration of Fig. 1.

4. CONCLUDING REMARKS

It is worth emphasizing that the choice of C_O^+ as the initial-datum hypersurface for $\phi_{AB\ldots C}(x)$ is what guarantees the viability of the prescription leading to (3.7). As far as the homogeneous structures are concerned, this fact clearly entails interchanging the roles played by the null slices of the projective twistor spaces when right-handed fields are explicitly taken into consideration. We should remark that the singularities of the integrand of equation (2.7) occur whenever the (null) geodesics representing $\{\overset{2}{W}_\alpha\}$ meet the generators associated with the auxiliary twistors involved. Under these circumstances, the point y turns out to be null-separated from the intersection points. Evidently, a similar observation is applicable to the integrand of equation (3.10) as well.

A striking feature of the techniques involved in the derivation of equation (3.9) is that the only parametrization procedure effectively allowed for is the one associated with the conformally invariant definition of vertices of null cones. At first sight, one might think of normalized integral kernels of the type

$$\frac{1}{2\pi i}\oint_{\gamma_1} \frac{(I^{\mu\nu}R_\mu S_\nu)\delta\overset{1}{W}}{(I^{\alpha\beta}\overset{1}{W}_\alpha R_\beta)(I^{\lambda\tau}\overset{1}{W}_\lambda S_\tau)} \tag{4.1}$$

as apparently natural auxiliary devices for obtaining the diagram expressed by (3.10). Nevertheless, this procedure would give rise to the "annihilation" of the combined integrand because its completion implies the occurrence of the identically vanishing piece

$$\overset{1}{W}_\mu \frac{\partial}{\partial\overset{2}{W}_\mu}\,\Phi(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) \tag{4.2}$$

Indeed, the above situation seems to bring out the adequacy and nontriviality of our methods. With respect to this fact, it should be observed that the connectedness of the diagrammatic blocks which emerge from the "splitting" of the functional structure of $F_s(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha)$ can be ensured if we look upon the infinity-twistor inner product carrying $\overset{1}{W}_\alpha$ and $\overset{2}{W}_\alpha$ as one of the singularity lines of the entire diagram. We point out that if the $\overset{1}{W}$-integration were somehow considered alone, then we would have to allow Poincaré-invariant branches to carry negative numbers upon opening up the bubble of Fig. 1. The condition for (3.4) to remain valid would thus amount to $I^{\mu\nu}Y_\mu \overset{2}{W}_\nu \neq 0$. However, this inner product must be regarded as a singularity of the overall integrand when the $\overset{2}{W}$-integration is put into effect. It follows that, writing

$$F_s(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) = \lambda_s \frac{(I^{\alpha\beta}J_\alpha Y_\beta)^{1-s}(I^{\gamma\delta}S_\gamma R_\delta)(I^{\mu\nu}\overset{1}{W}_\mu \overset{2}{W}_\nu)^{2s-1}}{(I^{\lambda\tau}\overset{1}{W}_\lambda R_\tau)(I^{\gamma\delta}\overset{1}{W}_\gamma S_\delta)(I^{\mu\nu}Y_\mu \overset{2}{W}_\nu)} \tag{4.3}$$

where

$$\lambda_s = 2\pi i(-1)^{-2s}(1 - 2s) \tag{4.4}$$

and representing by dashed-dotted lines the parametrization of $\overset{1}{W}_\alpha$ incorporated into (3.2) yields an equality carrying Fig. 1 along with the SI configuration of Fig. 3, with the thick-line branches particularly standing for the effective Poincaré-invariant singularities. Of course, the standard four-line rule is not applicable to the spotted vertices because of the pattern of the functional structure carried by $\Phi(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha)$. If the type of the denominator of the integrand of (3.1) had been modified, the construction of a diagram carrying a spotted $\overset{1}{W}$-vertex could also have been achieved.

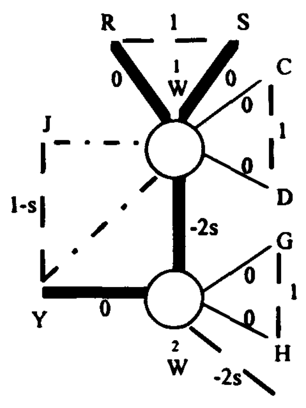

Fig. 3.

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